UDC 531.31

ON THE GENERALIZATION OF THE BONNET THEOREM

PMM Vol. 37, №6, 1973, pp. 1138-1141
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(Received December 7, 1972)

The well-known Bonnet theorem [1] is generalized to the case of motion of material points of variable mass moving under the action of a quasi-positional system of forces, i. e. of a system where each force is a function only of those parameters which determine the position of the point on the trajectory.

Let us consider a motion of each of n free material points of variable mass m_k taking place under the action of an active force $\mathbf{F}_k$ $(k = 1, \ldots, n)$, depending only on the position of the point on the trajectory [2]. We assume the mass of each point to be a continuously differentiable function of the curvilinear coordinate $m_k = m_k(s)$.

Each point of the mass m_k is acted upon by the reaction force $\mathbf{R}_k$. If we assume that

$$\mathbf{u}_k = \mathbf{w}_k(s) v_k^{-1} \tag{1}$$

where $\mathbf{u}_k$ is the absolute velocity of detachment (or attachment) of the particles and $\mathbf{w}_k$ is a continuous vector function, then the force $\mathbf{R}_k$ is quasi-positional

$$\mathbf{R}_k = \frac{dm_k}{ds} \mathbf{w}_k(s) \qquad (k = 1, \ldots, n) \tag{2}$$

Let the forces acting on the material points, and the initial conditions be such that each of these points describes one and the same trajectory, the curve AB. We shall determine the condition under which a material point of variable mass $M = M(s)$ describes the trajectoty AB, or at least a part of it. We assume $M(s)$ to be a continuously differentiable function.

Suppose that real numbers a_k and b_k exist such, that

$$\mathbf{F} = \sum_{k=1}^{n} a_k \mathbf{F}_k, \qquad \mathbf{R} = \sum_{k=1}^{n} b_k \mathbf{R}_k \tag{3}$$

where $\mathbf{F}$ and $\mathbf{R}$ are the forces acting on the point M. Following the terminology of [2] we shall call the motion caused only by forces $\mathbf{F}_k$ and $\mathbf{R}_k$, for each fixed k, the particular motion, while the motion caused by the forces $\mathbf{F}$ and $\mathbf{R}$, the composite motion.

Since we cannot postulate in advance that a point of mass M describes the curve AB during its composite motion, we introduce a restoring force $\mathbf{N}$ normal to this curve at every point [2] and such that the given point will describe the trajectory AB under the action of the force $\mathbf{F} + \mathbf{R} + \mathbf{N}$.

The equation of composite motion [3] has the form

$$d\mathbf{K}\,(dt)^{-1} = \sum_{k=1}^{n} a_k \mathbf{F}_k + \sum_{k=1}^{n} b_k \mathbf{R}_k + \mathbf{N}, \qquad \mathbf{K} = M\mathbf{v} \tag{4}$$

and yields

$$dT = \sum_{k=1}^{n} a_k \delta A_k(\mathbf{F}_k) + \sum_{k=1}^{n} b_k \delta A_k(\mathbf{R}_k) - \frac{1}{2} v^2 dM, \quad T = {}^1/_2 Mv^2, \qquad d\mathbf{r} = \mathbf{v}dt \quad (5)$$

where $\delta A_k(\mathbf{F}_k)$ and $\delta A_k(\mathbf{R}_k)$ denote the elementary work done.

Let $\mathbf{v}_k$ denote the velocities of the points of mass m_k in the particular motions. Then for the particular motions we have the following relations:

$$dT_k^* = \delta A_k(\mathbf{F}_k) - \frac{1}{2}(v_k^*)^2 dm_k^*, \quad T_k = \frac{1}{2} m_k v_k^2 \quad (6)$$

$$dT_k^\circ = \delta A_k(\mathbf{R}_k) - \frac{1}{2}(v_k^\circ)^2 dm_k^\circ, \quad d\mathbf{r}_k = d\mathbf{r} \quad (7)$$

In the formulas (6) and (7) and in what follows, the upper asterisk denotes the quantities referring to the particular motions of the points with mass m_k under the kinematic constraints of the form $\mathbf{w}_k = 0$ and the upper null index refers to the quantities in the partucular motions for the material points isolated from the active forces.

From the relations (6) and (7) we construct the basis combination

$$D = \sum_{k=1}^{n} a_k dT_k^* + \sum_{k=1}^{n} b_k dT_k^\circ \quad (8)$$

using now the fact that the quasi-positional active and reactive forces in the composite and particular motions are equal to each other and by virtue of the equation $d\mathbf{r}_k = d\mathbf{r}$ we obtain, from (5) and (8),

$$dT + \frac{1}{2} v^2 dM = \sum_{k=1}^{n} a_k dT_k^* + \sum_{k=1}^{n} b_k dT_k^\circ + \quad (9)$$

$$\frac{1}{2} \sum_{k=1}^{n} a_k (v_k^*)^2 dm_k^* + \frac{1}{2} \sum_{k=1}^{n} b_k (v_k^\circ)^2 dm_k^\circ$$

We assume that

$$v^2 dM = \sum_{k=1}^{n} a_k (v_k^*)^2 dm_k^* + \sum_{k=1}^{n} b_k (v_k^\circ)^2 dm_k^\circ \quad (10)$$

Integrating (9) and taking (10) into account, we obtain

$$T = T_0 + \sum_{k=1}^{n} a_k T_k^* + \sum_{k=1}^{n} b_k T_k^\circ \quad (11)$$

$$2T_0 = M_0 v_0^2 - \sum_{k=1}^{n} a_k m_{0k}^* (v_{0k}^*)^2 - \sum_{k=1}^{n} b_k m_{0k}^\circ (v_{0k}^\circ)^2 \quad (12)$$

where the zero subscript denotes the values of the quantities at $t = 0$.

Assuming that

$$M_0 v_0^2 = \sum_{k=1}^{n} a_k m_{0k}^* (v_{0k}^*)^2 + \sum_{k=1}^{n} b_k m_{0k}^\circ (v_{0k}^\circ)^2 > 0 \quad (13)$$

then $T_0 = 0$ and the quantity T is essentially positive at least in some half-neighborhood of the point A. We shall show that the restoring force $\mathbf{N} = 0$ at any interior point of the trajectory AB. Projecting both parts of Eq. (4) on the normal plane P of a natural trihedron, we obtain

$$Mv^2 \rho^{-1} \mathbf{n} = \sum_{k=1}^{n} a_k (\mathbf{F}_k)_P \sum_{k=1}^{n} b_k (\mathbf{R}_k)_P + N \quad (14)$$

where $\mathbf{n}$ is the unit vector of the principal normal, ρ is the radius of curvature of the trajectory and the subscript P denotes a projection of the vector on the respective plane.

For the particular motion we have

$$m_k^* (v_k^*)^2 \rho^{-1}\mathbf{n} = (\mathbf{F}_k)_P, \qquad m_k^\circ (v_k^\circ)^2 \rho^{-1}\mathbf{n} = (\mathbf{R}_k)_P \tag{15}$$

Having constructed the basis combination for (15), we use the fact that the forces $\mathbf{F}_k$ and $\mathbf{R}_k$ are identical in both the particular and the composite motions, and that $\mathbf{N} = N\mathbf{n}$, we obtain from (14) and (15)

$$T = \sum_{k=1}^{n} a_k T_k^* + \sum_{k=1}^{n} b_k T_k^\circ + \frac{1}{2}\rho N \tag{16}$$

Eliminating the values $\rho = 0$ and ∞ and equating the relations (16) and (11), we find that when the conditions (13) hold, $\mathbf{N} = 0$.

Thus we arrive at the following theorem:

T h e o r e m 1. Let every material point with quasi-positionally variable mass m_k $(k = 1, \ldots, n)$ describe one and the same trajectory AB, under the action of the quasi-positional active forces $\mathbf{F}_k$ and reactive forces $\mathbf{R}_k$. If the conditions (1), (3), (10) and (13) hold, a material point of mass M acted upon by the forces $\mathbf{F}$ and $\mathbf{R}$ and moving at the velocity $\mathbf{v}_0$ in the same direction as each of the $\mathbf{v}_{0k}$, describes at least a part of this trajectory adjacent to the point A.

The converse theorem can be shown to exist using the approach adopted in [2].

T h e o r e m 2. When the conditions of Theorem 1 hold and the curve AB is a trajectory of the material point of mass M moving at the velocity $\mathbf{v}_0$ under the action of forces (3), then the conditions (1), (10) and (13) hold along the curve AB.

Indeed, we can confirm that in this case the relations (14) and (15) also hold and, by the conditions of the theorm, $\mathbf{N} = 0$. Using the fact that the corresponding quasi-positional forces in (14) and (15) are equal to each other, we obtain an equality equivalent to the system of relations (11), (12).

Theorem 1 can be extended also to the case of a constrained motion when the trajectory AB lies on a smooth surface Q. The normal reaction forces $\mathbf{\Phi}$ and $\mathbf{\Phi}_k$ of the surface Q in the composite and particular motions are normal to the plane Q_τ tangent to the surface Q at the points lying on the curve AB, and the restoring force $\mathbf{N}$ can be chosen so that it lies in the plane Q_τ. Then the equation of composite motion is [3]

$$d\mathbf{K}\,(dt)^{-1} = \sum_{k=1}^{n} a_k \mathbf{F}_k + \sum_{k=1}^{n} b_k \mathbf{R}_k + \mathbf{N} + \mathbf{\Phi} \tag{17}$$

In this case (11) and (13) remain valid and in place of (15) we obtain

$$m_k^* (v_k^*)^2 \rho^{-1}\mathbf{n} = (\mathbf{F}_k)_P + \mathbf{\Phi}_k, \qquad m_k^\circ (v_k^\circ)^2 \rho^{-1}\mathbf{n} = (\mathbf{R}_k)_P + \mathbf{\Phi}_k \tag{18}$$

Projecting (17) on the plane P, we obtain

$$M v^2 \rho^{-1}\mathbf{n} = \sum_{k=1}^{n} a_k (\mathbf{F}_k)_P + \sum_{k=1}^{n} b_k (\mathbf{R}_k)_P + \mathbf{N} + \mathbf{\Phi} \tag{19}$$

If we construct, as before, the basis combination with the help of (18) and substitute it into the right hand side of (19), we obtain

$$2\left(T - \sum_{k=1}^{n} a_k T_k^* - \sum_{k=1}^{n} b_k T_k^\circ\right)\rho^{-1}\mathbf{n} = \mathbf{N} + \mathbf{\Phi} - \sum_{k=1}^{n} c_k \mathbf{\Phi}_k \quad (c_k = a_k + b_k) \qquad (20)$$

Since $\mathbf{N}$ is chosen orthogonal to $\mathbf{\Phi}$ and $\mathbf{\Phi}_k$, from (11) and (13) we find that $\mathbf{N} = 0$. Moreover, from Eq. (20) it then follows that

$$\mathbf{\Phi} = \sum_{k=1}^{n} c_k \mathbf{\Phi}_k$$

From the results obtained we have, as particular cases, the corresponding theorem for the points of constant mass [2] and the proper Bonnet theorem [1]. Just as it was done in [2], the results obtained can be applied to the study of motion of points of variable mass in a gravitational field of two fixed centers.

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Translated by L. K.

UDC 539.3

ON THE MEMBRANE STATE OF MULTIPLY-CONNECTED CONVEX SHELLS

PMM Vol. 37, №6, 1973, pp. 1141-1145

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(Received September 20, 1972)

Conditions for the realization of the membrane state of equilibrium of multiply-connected shells of positive Gaussian curvature subjected to surface and edge forces, are investigated ; the concepts of correctness and stability of the membrane states are introduced. The terminology and notation correspond to that used in [1, 2].

1. Let the middle surface S of a multiply-connected shell of positive Gaussian curvature be referred to an isometrically conjugate curvilinear coordinate system x^1, x^2 and let us write its equation in the vector form $\mathbf{r} = \mathbf{r}(x^1, x^2)$. Relative to the regularity of the shell we assume that $S \in D_{k+3,p}$, $p > 2$, $k \geqslant 0$. The middle surface S and its outline $L = L_0 + L_1 + \ldots + L_m$ in the coordinate plane $\zeta = x^1 + ix^2$ are a domain G with the boundaries $\Gamma = \Gamma_0 + \Gamma_1 + \ldots + \Gamma_m$ in a homeomorphic way. The lines of the holes in the shell $L_0, L_1, \ldots, L_m$ are closed, three-dimensional, nonreentrant curves of the Liapunov class. A x^1, x^2 coordinate system can always be found so that the point $\zeta = 0$ would belong to the interior of the domain G and the contour Γ_0 would enclose all the other curves $\Gamma_1, \ldots, \Gamma_m$. Finally, the second quadratic form of the